Bachelor of Science (B.Sc.) Semester—II Exam

MATHEMATICS

(M_4 : Vector Calculus and Improper Integrals)

Compulsory Paper—II

Time—Three Hours] [Maximum Marks—60

N.B. :— (1) Solve all the **FIVE** questions.

(2) All questions carry equal marks.

(3) Question No. **1** to **4** have an alternative. Solve each question in full or its alternative in full.

UNIT—I

1. (A) If $\vec{A} = \vec{A}(t) = 3t^2\,\vec{i} - (t+4)\,\vec{j} + (t^2 - 2t)\,\vec{k}$ and $\vec{B} = \vec{B}(t) = \sin t\,\vec{i} + 3e^{-t}\,\vec{j} - 3\cos t\,\vec{k}$, find $\frac{d^2}{dt^2}\left[\vec{A} \times \vec{B}\right]$ at $t = 0$. 6

(B) If $\vec{A} = 3xyz^2\,\vec{i} + 2xy^3\,\vec{j} - x^2yz\,\vec{k}$ and $\phi = 3x^2 - yz$, find (i) $\nabla \cdot \vec{A}$, (ii) $\vec{A} \cdot \nabla\phi$, (iii) $\nabla \cdot (\phi\vec{A})$ and (iv) $\nabla \cdot (\nabla\phi)$ at the point $(1, -1, 1)$. 6

OR

...ny ...d constants a, b, c so that

$W\vec{v} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$

at... is irrotational. 6

(D) Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\,\vec{i} - 5z\,\vec{j} + 10x\,\vec{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$. 6

UNIT—II

2. (A) Show by double integration that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $16\dfrac{a^2}{3}$. 6

(B) Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{(a^2 - y^2)}} \sqrt{\left(a^2 - x^2 - y^2\right)}\,dydx$. 6

OR

(C) Evaluate $\displaystyle\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dxdy$ by changing the order of integration. 6

(D) Show that :

$$\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2\,dx\,dy\,dz = \frac{a^5}{60}.$$ 6

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Contd.

UNIT—III

(A) Evaluate $\iint_S \vec{A} \circ \vec{n}\, dS$, where $\vec{A} = y\vec{i} + 2x\vec{j} - z\vec{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$. 6

(B) By using Green's theorem, prove that the area bounded by a simple closed curve C is given by $1/2 \oint_C (xdy - ydx)$. Also in polar coordinates (ρ, ϕ), show that $xdy - ydx = \rho^2 d\phi$. 6

OR

(C) Evaluate $\iint_S \vec{F} \circ \vec{n}\, dS$ by using the divergence theorem, where $\vec{F} = 2xy\,\vec{i} + yz^2\,\vec{j} + xz\,\vec{k}$ and S is the surface of the parallelopiped bounded by $x = 0$, $y = 0$, $z = 0$, $x = 2$, $y = 1$ and $z = 3$. 6

(D) Verify Stoke's theorem for $\vec{F} = xz\,\vec{i} - y\,\vec{j} + x^2y\,\vec{k}$, where S is the surface of the region bounded by $x = 0, y = 0, z = 0, 2x + y + 2z = 8$ which is not included in the xz-plane. 6

UNIT—IV

4. (A) Prove that $\int_a^\infty \frac{dx}{x^p}$, where p is a constant and $a > 0$, converges if $p > 1$ and diverges if $p \le 1$. 6

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(B) Test for convergence of $\int_2^\infty \frac{x\,dx}{\sqrt{(x^3-1)}}$ an

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{\left(x^2+x+1\right)^{5/2}}.$$

OR

(C) Prove that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, where 'n

is a positive integer and $m > -1$. 6

(D) Given $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$, show tha

$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$, where $0 < p < 1$. Hence

or otherwise show that $\int_0^\infty \frac{dy}{1+y^4} = \frac{\pi\sqrt{2}}{4}$. 6

UNIT—V

5. (A) Evaluate $\operatorname{div}\left[2x^2 z\,\bar{i} - xy^2 z\,\bar{j} + 3yz^2\,\bar{k}\right]$. 1½

(B) Determine the constant 'a' so that the vector $\vec{v} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+az)\bar{k}$ is solenoidal. 1½

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(C) Evaluate $\int_{0}^{\pi/2} \int_{0}^{a\cos\theta} dr\, d\theta$. 1½

(D) Evaluate $\int_{0}^{1} \int_{0}^{2} \int_{0}^{3} z\, dx\, dy\, dz$. 1½

(E) If $\oint_{C} M\, dx + N\, dy = 0$ around every closed curve C in a simply connected region, then prove that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ everywhere in the region. 1½

(F) Evaluate $\iint_{S} \bar{r} \circ \bar{n}\, dS$ by using divergence theorem, where S is a closed surface. 1½

(G) Evaluate $\int_{0}^{1} \sqrt{\left[\frac{(1-x)}{x}\right]}\, dx$. 1½

(H) Evaluate $\int_{0}^{\infty} x^{6} e^{-3x}\, dx$. 1½